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On the Measure of Fuzziness and Negation. II. Lattices

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We propose that the measure of fuzziness of a concept is related to the distinction between the concept and its negation. In this paper we are concerned with the situation in which our objects lie in a lattice. We introduce the lattice concept of betweenness to operationalize our definition. We investigate properties of this concept of fuzziness. We also discuss the related notion of negation in lattices in detail.

INTRODUCTION

The problem of defining and measuring fuzziness or imprecision is an important question. In 1965, Zadeh introduced fuzzy sets and in doing so developed a structure on which a quantitative study of fuzziness can be attempted. His initial paper has generated a large number of papers on fuzzy sets (Kandel and Yager, 1979). DeLuca and Termini (1972) introduced the concept of non-probabilistic entropy in an attempt to measure fuzziness. That paper has itself generated a number of papers in that area. (See DeLuca and Termini (1974), Capocelli and DeLuca (1972), Knopfmacher (1975), Pollatschek (1977), DeLuca and Termini (1977), Loo (1977).)

Yager (1979) has suggested an intuitive definition of fuzziness. He associates fuzziness with the lack of distinction between a proposition and its negation, specifically the further a concept is from its negation the less fuzzy it is. Conversely, if in the concept space an idea and its negation are close the concept is said to be more fuzzy. He then proceeds to quantify this idea for fuzzy subsets, where the grades of membership lie in the unit interval. In particular, he uses measures of distinction based on metric distances between fuzzy sets and on compatibility of fuzzy sets.

In this paper we shall generalize the work of Yager (1979) by attempting to investigate the ramifications of Yager's definition of fuzziness to situations in which we are dealing with lattices and L-fuzzy sets. In particular, we operationalize the definition of fuzziness in lattices by introducing the lattice concept of betweenness to measure distinctions between elements in lattices. We shall concern ourselves with the implications for measuring fuzziness under various assumptions about the negation defined on the lattice. We discuss order-reversing

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involutionary negations and intuitionistic negations (Heything (1956)). We shall see that the concept of fuzziness of an object leads to a measurement of fuzziness which is a partial ordering over the meet of an element and its negation. A detailed discussion of negation is presented since negation is intimately connected with fuzziness.

Fuzzy Sets and Lattices

In this paper we consider fuzziness as being the lack of distinction between a proposition, concept, or word and its negation. In a number of papers Zadeh (1975, 1976) has shown that the structure of fuzzy sets presents and ideal medium in which to study logic and linguistics. Thus, by using the fuzzy sets to represent propositions, concepts, and words we can handle a large class of situations in which fuzziness appears. We shall attempt to study fuzziness by using fuzzy sets to represent the concepts whose fuzziness we want to measure.

The basic elements of fuzzy sets, as introduced by Zadeh (1965) and generalized by Goguen (1967, 1969), consist of a set of elements X, a set S of grades of membership, and a class of fuzzy subsets of X. These fuzzy subsets are characterized by assigning to each $x \in X$ a value $s \in S$ indicating the grade of membership of x in the fuzzy subset.

If A is a fuzzy subset of X then

$$A: X \rightarrow S \qquad \text{where} \quad A(x) = s.$$

In addition, we define two binary operations and one unary operation on the class of fuzzy subsets. These operations are, respectively, union, intersection, and negation. It is generally the case that these operations are defined pointwise on the fuzzy sets. By this we mean that if A and B are any two fuzzy subsets of X, with $x \in X$ and if

$$C = A * B$$

then

$$C(x) = f[A(x), B(x)].$$

That is, $C(x)$ just depends upon A and B at x. Similarly if $\sim$ is the unary operation on A, then $\sim A(x) = G[A(x)]$.

We shall use the following notation to represent the operations on fuzzy subsets. For union

$$A \cup B = C,$$

where the pointwise operation is defined

$$C(x) = A(x) \vee B(x),$$

and for intersection

$$A \cap B = D, \qquad \text{where} \quad D(x) = A(x) \wedge B(x).$$

In the above, $\wedge$ and $\vee$ are the operations meet and join.

Since fuzzy sets are a generalization of ordinary sets we associate with the operations of intersection and union certain properties associated with the operations on ordinary sets. The properties which we desire are idempotency, commutativity, associativity, and absorption. The pointwise definition of these operations has been shown to imply (Goguen, 1967, 1969); Kaufman, 1975) that these properties, idempotency, etc., must also be associated with the meet and join operations for the set S of grades of membership. It is well known that a set having two binary operations satisfying the four listed properties is a lattice structure. Thus both the class of fuzzy subsets of X and the set of membership grades S have a minimal structure of a lattice. Since we are using fuzzy subsets to describe fuzziness and the class of these subsets form a lattice we can now simply study the measures of fuzziness in a lattice. The fact that both grades of membership and the class of fuzzy subsets have structures of a lattice gives us two courses of action for the measures of fuzziness which we shall find for elements of a lattice.

In the first case we can consider our lattice as consisting of the fuzzy subsets and then apply our measures directly to these subsets. In the second approach we can consider our measures of fuzziness defined on the lattice of membership grades and then combine the fuzziness measures for the grades making up the particular subset to give us the fuzziness measure for that fuzzy subset.

In this paper we shall not make a distinction between the two approaches; we shall just consider fuzziness in lattices. So, when we consider a lattice L its members could be fuzzy subsets or grades of membership and hence we shall from here on be concerned with the measurement of fuzziness in a lattice.

A lattice consists of a set L of elements and two operations, $\wedge$ and $\vee$, which satisfy the following properties for $x, y, z \in L$;

(1) idempotency; $x \wedge x = x$ and $x \vee x = x$,

(2) commutativity; $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$,

(3) associativity; $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$,

(4) absorption; $x \wedge (x \vee z) = x = x \vee (x \wedge y)$.

A lattice is said to be distributive if it satisfies the distributivity property,

(5) distributivity;

(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge y)$,

(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

It should be noted that conditions i and ii are equivalent.

A lattice is said to possess universal bounds if it contains two special elements 0 and I s.t. for all $x \in L$

$$(6) \quad \begin{aligned} x \wedge 0 &= 0, \\ x \wedge I &= x. \end{aligned}$$

It can be shown that the unit interval is a distributive lattice with $\wedge = \min$, $\vee = \max$, and 0 and 1 the universal bounds. (Birkoff (1968).)

A simple property which a lattice is seen to have is that of consistency; this implies

$$x \wedge y = x \quad \text{iff} \quad x \vee y = y.$$

An immediate corollary to this property is the fact that

$$0 = x \wedge 0 \Rightarrow x \vee 0 = x,$$
$$x = x \wedge I \Leftrightarrow x \vee I = I.$$

We can define a binary a relation $\leqslant$ on our lattice L as

$$x \leqslant y \quad \text{iff} \quad x \wedge y = x \ (\text{or } x \vee y = y).$$

It can be shown that the binary relation $(\leqslant, L)$ is a poset on L, (Birkhoff (1968)). That is, this relation is reflexive, antisymmetric, and transitive. If for all x and y

$$x \wedge y \in \{x, y\},$$

we say our lattice is totally ordered or a chain.

Fuzziness in Lattices

Having presented the fundamental properties of lattices we shall now address our main concern, the concept of fuzziness and the related idea of negation.

We shall assume that we have a lattice L with a binary operation called negation. That is,

$$N: L \to L$$
$$\text{s.t.} \quad N(x) \to x^*, \quad \text{for} \quad x \in L,$$

where x^* is called the negation of x. Subsequently, we shall go into more detail about the properties and procedures for defining this operation. At this point all that is needed is an intuitive idea of negation to help us understand the concept of fuzziness in a lattice. As discussed in the previous part of this paper the measure of fuzziness of a fuzzy subset is related to the distinction or distance between a subset and its negation. In particular, we showed that the closer a subset is to its negation the fuzzier the subset. Thus, based upon the idea that the further the two are from each other the clearer the associated concept, we should now like to define a measure of fuzziness for elements in a lattice. In particular, we should like to associate with each element $x \in L$ a measure of its fuzziness. This measure of fuzziness, associated the element x, should be related to the distance between x and its negation x^*. That is, the closer x and x^* are in the lattice the more fuzzy or less distinct x. The problem we are faced with then is that of obtaining some measure of distance in a lattice. We shall use the idea of betweeness to give us this measure.

Birkoff (1968) defines the concept of betweenness in a lattice as follows:

DEFINITION 1. In a lattice we write (a, b, c) indicating b is between a and c iff

$$(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (b \vee c).$$

It can be easily shown, using the commutative property of lattices, that

$$(a, b, c) \Leftrightarrow (c, b, a).$$

This implies that betweenness is independent of the ordering of a and c. This concept enables us to indicate an element of L as being between two elements regardless of the order of the two bounding elements.

In a distributive lattice, the betweenness condition becomes

$$b \wedge (a \vee c) = b = b \vee (a \wedge c).$$

One can easily verify that in a lattice L with universal bounds all elements in L between these bounds. That is, for any $x \in L$

$$(O \wedge x) \vee (x \wedge I) = O \vee x = x,$$
$$(0 \vee x) \wedge (x \vee I) = x \wedge I = x.$$

Thus, (O, x, I), which also implies (I, x, O).

A number of useful properties of betweenness have been developed in the literature (Birkoff, 1968; Smiley and Transue, 1943; Sholander, 1952, 1954a, 1954b). A particularly interesting discussion of the relationship between metrics in lattices and betweenness is presented in Smiley and Transue (1943). We shall list some of the properties of (a, b, c) which we shall find useful.

P_1 $(abc) \Leftrightarrow (cba)$.
P_2 (abc) and $(acb) \Leftrightarrow b = c$.
P_3 (abc) and $(axb) \rightarrow (axc)$.
P_4 (abc), (bcd) and $b \neq c \rightarrow (abd)$.
P_5 (abc) and $(acd) \rightarrow (bcd)$.
P_6 $(aba) \Leftrightarrow a = b$.
P_7 $(aab) \Leftrightarrow (baa) \Leftrightarrow (abb) \Leftrightarrow (bba)$.
P_8 $(abc) \rightarrow (aab)$.

P_9 If L is a distributive lattice

(a) (pbc), (pdc), and $(bxd) \rightarrow (pxc)$,

(b) (pbc), (pdb) and $(cxd) \rightarrow (pbx)$.

P_{10} $(abc) \rightarrow a \wedge c \leqslant b \leqslant a \vee c$.

We shall use this concept of betweenness to define the fuzziness of an element in a lattice. Recall that we stated that x is said to be fuzzier than y if x and x^* are closer in the lattice than y and y^*. We can use the concept of betweenness to operationalize this idea.

DEFINITION 2. Given two elements x and y in a lattice L, we shall say that element x is at least as fuzzy as y, denoted xfy, if the following two conditions hold:

(1) (y, x, y^*),

(2) (y, x^*, y^*).

That is, x is said to be fuzzier than y if x and x^* lie between y and y^*. In the above definition we have defined fuzziness as a relationship on the lattice L. We shall say that x and y are equally fuzzy, denoted xIy, if xfy and yfx.

Since f is a relationship on the lattice L we can investigate which properties of relationships f satisfies.

THEOREM 1. *f is reflexive on the lattice L, xfx.*

Proof.

$(x \wedge x) \vee (x \wedge x^*) = x \vee (x \wedge x^*) = x$ from absorption on L.

$(x \vee x) \wedge (x \vee x^*) = x \wedge (x \vee x^*) = x$ from absorption property.

Thus (x, x, x^*). Similarly we can show that (x, x^*, x^*).

THEOREM 2. *If L is also a distributive lattice, then f is transitive. That is, xfy, $yfz \rightarrow xfz$.*

Proof.

$$\left.\begin{array}{l} yfz \rightarrow (z, y, z^*) \text{ and } (z, y^*, z^*) \\ xfy \rightarrow (y, x, y^*) \text{ and } (y, x^*, y^*) \end{array}\right\} \Rightarrow$$

$$(z, y, z^*) \text{ and } (z, y^*, z^*) \text{ and } (y, x, y^*) \Rightarrow (z, x, z^*).$$

This follows from property 9a. Similarly replacing x and x^* implies (z, x^*, z^*). Therefore xfz.

THEOREM 3. *If xfy and yfx, then either $x = y$ and $x^* = y^*$ or $x = y^*$ and $y = x^*$.*

Proof.

$$xfy \Rightarrow (y, x, y^*) \text{ and } (y, x^*, y^*),$$
$$yfx \Rightarrow (x, y, x^*) \text{ and } (x, y^*, x^*).$$

Assume $x \neq y$; then

$$(y^*, x, y) \text{ and } (x, y, x^*) \text{ and } x \neq y \rightarrow (y^*, x, x^*) \Leftrightarrow (x^*, x, y^*) \text{ from } P_4.$$

From P:

$$(x^*, x, y^*) \text{ and } (x^*, y^*, x) \rightarrow y^* = x.$$

Furthermore, since $y^* = x$ then

$$(x, y, x^*) = (y^*, x, x^*).$$

However, (y^*, y, x^*) and (y^*, x^*, y) from P_2

$$x^* = y.$$

On the other hand if $x = y$ then

$$(x, y^*, x^*) = (y, y^*, x^*).$$

However, (y, y^*, x^*) and (y, x^*, y^*), from P_2

$$x^* = y^*.$$

It should be noted that Theorems 1–3 assumed no particular properties about negations; they are valid however we define negation. However, Theorem 3 does not imply that f is antireflexive over L, and hence f is not yet a partial ordering.

Negation and Fuzziness

The question naturally arises concerning the properties of a negation in a lattice. Lowen (1978) suggests that the negation in a lattice used to carry fuzzy subsets should have the following two properties:

(1) Negation should be an involution.

(2) Negation should be order reversing.

Goguen (1969) has also suggested that this is one possibly way of describing negation. An alternative procedure involves using the Browerian implication (Goguen, 1969, Birkoff, 1968). We shall first investigate the situation for a negation satisfying the two conditions suggested by Lowen.

DEFINITION 3. Assume L is a lattice. Then a unary operation,

$$N: L \to L$$

s.t.

$$N(x) = x^*,$$

where

(1) $(x^*)^* = x$,

(2) $(x \wedge y) = z \Rightarrow x^* \vee y^* = a^*$
$(x \vee y) = a \Rightarrow x^* \vee y^* = z^*$

is called an order-reversing involutionary negation.

It should be noted that condition 2 is the order-reversing property generalized to an arbitrary lattice.

THEOREM 4. *If L is a lattice with an order-reversing involutionary negation, denoted $N(x) = x^*$, then*

(1) *DeMorgan's laws are satisfied:*

$$(x \wedge y)^* = x^* \vee y^*,$$
$$(x \vee y)^* = x^* \wedge y^*.$$

(2) *If L, in addition, has universal elements, 0 and I, then*

$$I^* = 0 \qquad \textit{and} \qquad 0^* = I.$$

Proof.

(1) $x \wedge y = z \Rightarrow x^* \wedge y^* = q^*$,
$x \vee y = q \Rightarrow x^* \vee y^* = z^*$

then $(x^* \vee y^*)^* = (z^*)^* = z$ from involution.

Since $x \wedge y = z$, then $(x \wedge y) = (x^*y^*)^*$. However, since $((x^* \vee y^*)^*)^* = x^* \vee y^*$ then $(x \wedge y)^* = x^* \vee y^*$.

(2) Since DeMorgan's law and involution imply $x \vee y^* = (x^* \wedge y)^*$, if $y = I$, then $(x \vee I^*) = (x^* \wedge I)^* = (x^*)^* = x$. Since $(x \vee I^*) = x$ for all x, from the uniqueness of universal bounds we get $I^* = 0$.

Then we see $(I^*)^* = I$ and $(I^*)^* = 0^*$; we see $0^* = I$.

If the negation is order reversing and involutionary then we can cut in half the amount of work necessary to describe fuzziness.

THEOREM 5. *Assume L is a lattice with an involutionary negation satisfying DeMorgan's laws. Then*

(1) xfy if $(x \vee y) \wedge (x \vee y^*) = x$ and $(x^* \vee y) \wedge (x^* \vee y^*) = x^*$ or

(2) xfy if $(x \wedge y) \vee (x \wedge y^*) = x$ and $(x^* \wedge y) \vee (x^* \wedge y^*) = x^*$.

Proof. (1) From DeMorgan's law, $(x \vee y) \wedge (x \vee y^*) = x$ implies $(x \vee y)^* \vee (x \wedge y^*)^* = x^*$, which implies $x^* = (x^* \wedge y^*) \vee (x^* \wedge y)$. Similarly, $(x^* \vee y) \wedge (x^* \vee y^*) = x^* \rightarrow x = (x^* \vee y)^* \vee (x^* \vee y^*)^* \rightarrow x = (x \wedge y^*) \vee (x \wedge y)$. Thus the conditions of our definition are satisfied.

(2) Similarly proven.

Since order reversal and involution imply DeMorgan's law, the following corollary immediately follows.

COROLLARY. *If L is a lattice with an order-reversing involutionary negation then conditions* (1) *and* (2) *of the above theorem also hold.*

We shall now investigate some of the implications of the involution property of negation with respect to our measuring fuzziness of elements in a lattice.

DEFINITION 4. If the negation x^* of every element x in a lattice satisfies the property

$$(x^*)^* = x$$

then the negation operation is said to be an involution.

A fundamental implication of involution in a lattice is that the negation operation is a one-to-one mapping.

THEOREM 6. *If L is a lattice with an involutionary negation than the negations are unique $x \neq y \leftrightarrow x^* \neq y^*$.*

Proof. Assume $x \neq y$ and $x^* = y^* = a$. Then $a^* = (x^*)^* = x$, $a^* = (y^*)^* = y$, $y = x$ a contradiction.

The implication of this theorem is that we can consider an element and its negation as a unique pair.

If L is a lattice having negation we can define a relation E on the lattice such that, xEy if $x = y$ or $x^* = y$ or $x = y^*$. If the negation has the involution property then E becomes an equivalence relation. Thus, with the negation having involution we can decompose L into sets such that each element appears in one and only one set. Each of these equivalence classes or sets consists of an element and its negation. In general these classes consist of two elements. When $x = x^*$ then the equivalence class consists of only one element.

DEFINITION 5. We shall denote a subset of L consisting of an element and its negation as a negate or negate pair.

If $\mathscr{L}$ is the set whose elements are the equivalence classes of L under E, each element of $\mathscr{L}$ is a negate pair and hence the elements of $\mathscr{L}$ are subsets of L consisting of an element and its inverse.

THEOREM 7. *Assume L is a lattice which has a negation having the property of involution. Then*

(1) *x^*fx and xfx^*,*

(2) *$xfy \Leftrightarrow x^*fy \Leftrightarrow x^*fy^* \Leftrightarrow xfy^*$.*

Proof. (1) First we shall show that xfx^*; this requires $(x^*, x, (x^*)^*)$ and $(x^*, x^*, (x^*)^*)$. However, by involution $(x^*)^* = x$; thus we need (x^*, x, x) and (x^*, x^*, x). To show (x^*, x, x) we see $(x^* \wedge x) \vee (x \wedge x) = (x^* \wedge x)\, v(x) = x$ from absorption and $(x^* vx) \wedge (xvx) = (x^* vx) \wedge (x) = x$ from absorption. Furthermore, property 7 states $(x^*, x, x) \Rightarrow (x^*, x^*, x)$, therefore we have shown that xfx^*. To show x^*fx requires (x, x^*, x^*) and $(x, (x^*)^*, x^*)$, which again follow from property 7.

(2) Assume xfx.

$$xfy \Rightarrow (y, x, y^*) \text{ and } (y, x^*, y^*).$$

Since $(x^*)^* = x$, then $(y, x, y^*) \Rightarrow (y, (x^*)^*, y^*)$. However, (y, x^*, y^*) and $(y, (x^*)^*, y^*) \Rightarrow x^*fy$. For x^*fy^* we need $(y, {}^*x^*, y^{**})$ and (y^*, x^{**}, y^{**}). However, from involution $(y^*, x^*, (y^*)^*) = (y^*, x^*, y)$ and (y^*, x, y), both of which are implied by xfy.

In a similar manner we can show the rest of the equivalences. The implication of this theorem is that we can obtain a partial ordering over the negate pairs, the set $\mathscr{L}$.

COROLLARY 1. *Assume u and v are negates; that is, $u, v \in \mathscr{L}$. Assume $x, y \in L$ are such that $x \in u$ and $y \in v$. Then xfy implies zfw for all $z \in u$ and $w \in v$.*

That is, if one element in a negate pair is fuzzier than an element in a second negate pair, then any element in the first pair is fuzzier than any element in the second pair. Thus we can meaningfully define fuzziness of negate pairs.

DEFINITION 6. Assume L is a lattice with negation having the involution property. If $\mathscr{L}$ is the set of negates of L then we can define a relation F on $\mathscr{L}$, where we state that for u and $v \in \mathscr{L}$,

$$uFv$$

if xfy for any $x \in u$ and $y \in v$. If uFv say that the negate pair u is fuzzier than the negate v. It should be recalled that x and y in the above are contained in L.

Thus, we have extended the concept of fuzziness in lattices with involution to define a relationship on the set $\mathscr{L}$ of negate pairs. It should be noted the above corollary implies that if one negate pair is fuzzier than a second negate pair then each element in the first is fuzzier than each element in the second. Thus, there is unique correspondence between F and f in that F defines f uniquely, and vice versa. However, by considering negate pairs instead of elements, since our relationship F has the property of antisymmetry, we obtain a partial ordering.

THEOREM 8. *Assume L is a distributive lattice with negation having the involution property. Then, the relationship F defined over the set of negate pairs $\mathscr{L}$ is a partial ordering.*

Proof. We must show that F is reflexive, antisymmetric, and transitive.

(1) Reflexive requires uFu; consider the element $x \in u$, since xfx then uFu.

(2) Transitive requires uFv and $vFw \Rightarrow uFw$; uFv implies xfy for all $x \in u$ and $y \in v$; vFw implies yfz for all $y \in v$ and $z \in w$;

however, from Theorem 2 xfy and $yfz \rightarrow xfz$, which implies uFw.

(3) Antisymmetry requires uFv and $vFu \rightarrow u = v$.

$$uFv \rightarrow \text{if } x \in u \text{ and } u \in v \text{ then } xfy,$$
$$vFu \rightarrow \text{if } x \in u \text{ and } y \in v \text{ then } yfx.$$

From Theorem 2 we note that xfy and $yfy \Rightarrow x = y$ or $x = y^*$; however, if $x = y$ or $x = y^*$ then $u = v$.

Thus, we have shown that in a distributive lattice with a negation satisfying involution we can study the concept of fuzziness using negate pairs. Furthermore, the measure of fuzziness on negate pairs has the property of a partial ordering. Thus on the set $\mathscr{L}$ of negate pairs, we can define a lattice structure using the partial ordering F.

A property sometimes associated with the concept of negation in lattices with maximal and minimal elements is complementation.

We shall now show that any negate pair which is also complemented is a minimal element. That is, a complemented pair is least fuzzy.

DEFINITION 7. If $x \in L$ an element $\tilde{x}$ is called its complement if $x \wedge \tilde{x} = 0$ (law of contradiction) and $x \vee \tilde{x} = I$ (law of the excluded middle).

If x has a complement it is called complemented. If every element in a lattice is complemented then the lattice is said to be complemented. In some lattices the negation of an element is its complement.

THEOREM 9. *If x is an element in a distributive lattice L whose negation is its complement then yfx for all $y \in L$.*

Proof.

$$y \wedge (x \vee x^*) = y \wedge I = y,$$
$$y \wedge (x \wedge x^*) = y \vee 0 = y.$$

Similarly, we see that

$$y^* \wedge (x \vee x^*) = y^* \wedge I = y^*,$$
$$y^* \vee (x \wedge x^*) = y^* \vee 0 = y^*.$$

This theorem implies that if the negation of an element in a distributive lattice satisfies the property of complementation then this element is a least-fuzzy element.

There are certain properties of complementation we shall find useful.

THEOREM 10. *Assume L is a lattice with maximal and minimal elements I and* 0. *Assume the complement of x, if it exists, is denoted* $\tilde{x}$.

c1. $\tilde{x} \neq x$,

c2 *The complement is unique.*

c3 $(\tilde{\tilde{x}}) = x$.

c4 $\tilde{I} = 0$ *and* $\tilde{0} = I$.

Proof. c1. Assume $x = \tilde{x}$.

$$\tilde{x} \wedge x = x \wedge x = x = 0,$$
$$\tilde{x} \vee x = x \vee x = x = I,$$

a contradiction.

c2. Assume a and b are complements of x.

$$x \wedge a = 0 \qquad x \vee a = I,$$
$$x \wedge v = 0 \qquad x \vee b = I,$$
$$b \vee (x \wedge a) = b \vee I = b,$$
$$b \vee (x \wedge a) = (b \vee x) \wedge a = I \wedge a = a \rightarrow b = a.$$

c3. $x \vee \tilde{x} = I$ then $\tilde{x} \vee x = I$

$$\Rightarrow (\tilde{\tilde{x}}) = x.$$

$x \wedge \tilde{x} = 0$ $\qquad$ $\tilde{x} \vee x = 0$

c4. $I \wedge 0 = 0$ and $I \vee 0 = I$.

We note that the complement has the involution property.

THEOREM 11. *If L is a distributive lattice, with maximal and minimal elements and a negation operation having the involution property then:*

(1) *If u is a negate pair having the complement property than, u is a minimal element of the partial ordering over $\mathscr{L}$. That is, the complemented negate pair is least fuzzy.*

(2) *If u and v are negate pairs both having the complementation property than they are equally fuzzy.*

Proof. (1) Let $x \in U$, since from Theorem 4 yfx for all $y \in L$. This implies vFu for $v \in \mathscr{L}$. Thus u is a minimal element.

(2) Let $x \in u$ and $y \in v$, since from Theorem 4 xfy and yfx. Then uFv and $vFu \Rightarrow$ they are equally fuzzy.

THEOREM 12. *If L is a distributive lattice with negation having the property of complementation for all the elements then each negate pair is equally fuzzy.*

Proof. From property c3 complementation has the involution property. Therefore, from Theorem 11, all the negate pairs are equally fuzzy.

Thus, in a complemented lattice the concept of fuzziness does not exist. We should also note that Theorem 11 implies that any complemented pair in a lattice is the least-fuzzy negate pair.

THEOREM 13. *In a Boolean algebra no concept of fuzziness exists.*

Proof. A Boolean algebra has a negation which is complemented and satisfies involution.

The opposite of the idea of complemented negate is the self-negate.

DEFINITION 8. An element x in a lattice L with negation is called self-negate if $x^* = x$.

We note that if negation has the involution property then the equivalence class corresponding to x consists of one element. Thus, the negate pair of a self-negate element is one element. We also note that, from Theorem 10, a complemented element cannot be a self-negate, and vice versa.

THEOREM 14. *If L is a lattice with negation and if $x^* = x$ then there exists no $y \in L$ such that yfx when $x \neq y$ or $x \neq y^*$.*

Proof. Assume yfx. This implies (x, y, x^*) and (x, y^*, x^*). However, since $x = x^*$ then $(x, y, x^*) = (x, y, x)$ and therefore from P_6 $x = y$. Similarly for $y^* = x$.

This theorem, which is valid for any lattice with negation, implies that there is no element fuzzier than a self-negated element.

COROLLARY 1. *If $x, y \in L$, where $x^* = x$ and $y^* = y$ and $y \neq x$ then x and y are incomparable with respect to fuzziness. That is, neither xfy or yfx.*

COROLLARY 2. *If L is a lattice having an involutionary negation and if $u \in \mathscr{L}$ is a self-negate then vfu for all $v \in \mathscr{L}$.*

COROLLARY 3. *If L is a distributive lattice with a negation having the involution property and if there exists more than one distinct element in L which are self-negate then F on $\mathscr{L}$ cannot be a total ordering.*

We shall now investigate the effects of order reversal on fuzziness.

THEOREM 15. *If L is a lattice with an order-reversing negation then*

$$x \geqslant y \Rightarrow y^* \geqslant x^*.$$

Proof. For $x \geqslant y$

$$\begin{matrix} (x \wedge y) = y \\ (x \vee y) = x \end{matrix} \Rightarrow \begin{matrix} x^* \wedge y^* = x^* \\ x^* \vee y^* = y^* \end{matrix}$$

so that $y^* \geqslant x^*$. Note that in this theorem involution of negation is not needed.

THEOREM 16. *Assume L is a lattice with a negation satisfying the order-reversing property. If x is a self-negate then for any $y \in L$ s.t. $x \wedge y \in \{x, y\}$, xfy.*

Proof. Assume $x \wedge y = y$; then from the order-reversing property $x^* \wedge y^* = x^*$ we have $y \leqslant x = x^* = y^*$. Furthermore,

$$(x \wedge y) \vee (x \wedge y^*) = y \vee x = x,$$
$$(x \vee y) \wedge (x \wedge y^*) = x \wedge y^* = x.$$

If $x \vee y = y$ then we know $y^* \leqslant x = x^* \leqslant y$, which gives the same results. Thus in a lattice having an order-reversal property any element comparable with a self-negate is less fuzzy.

THEOREM 17. *Assume L is a lattice with a negation satisfying the order-reversing property. If L has maximal and minimal elements I and 0 then*

$$xfI \text{ and } xf0, \qquad \text{for all} \quad x \in L.$$

Proof.

$$(x \wedge I) \vee (I^* \wedge x) = x \vee I^* = x,$$
$$(x \vee I) \wedge (I^* \vee x) = x \wedge I = x.$$

Similarly for x^* and for 0.

We shall now investigate the concept of fuzziness and the related idea of negation in a totally ordered lattice.

THEOREM 18. *Assume L is a totally ordered lattice with a negation satisfying the order-reversal property then there exists at most one distinct element that is self-negated.*

Proof. Assume $x^* = x$ and $y^* = y$. Assume $y \neq x$ with no loss of generality, assume $y > x$; then $x \vee y = x$ and from Theorem 15 $x^* \wedge y^* = y^*$. However, since $x = x^*$ and $y = y^*$, then $x^* \wedge y^* = x \wedge y \Rightarrow x = y^*$. However, this contradicts our hypothesis $y \neq x$.

If L is a lattice which has a negation which is an involution we have shown we could decompose L into unique pairs of negates. This decomposition led us to the set $\mathscr{L}$ of equivalence classes under this negation. If L is also totally ordered, then we can order the elements in each equivalence class by this ordering. In particular, in each equivalence class u we shall denote the minimal element as u and the maximal element as u^*. Thus, if $\{a, b\}$ form an equivalence class under negation and if $a \wedge b = a$, then we shall denote $b = a^*$ and let a indicate the element in the set $\mathscr{L}$. Thus, in $\mathscr{L}$, $a = \{a, a^*\}$, where $b = a^*$.

LEMMA. *Assume L is a totally ordered lattice with an order-reversing involutionary negation. If a, $b \in \mathscr{L}$, that is, a and b are the minimal elements of their respective negate pairs, and if $a \leqslant b$ then*

$$a \leqslant b \leqslant b^* \leqslant a^*.$$

Proof.

$$a \leqslant b \Rightarrow a \wedge b = a.$$

By definition $a \wedge a^* = a$ and $b \wedge b^* = b$, from the order-reversal property of negation $a \leqslant b \Rightarrow b^* \leqslant a^* \Rightarrow a^* \wedge b^* = b^*$. Thus

$$a \leqslant b \leqslant b^* \leqslant a^*.$$

An immediate corollary to this lemma is that if L is a lattice which is totally ordered and has an involutionary order-reversing negation, then if s is a self-negated element

$$x \leqslant s \leqslant x^* \qquad \text{for all} \quad x \in L.$$

Thus each negate pair has one element above and below a self-negate, since in a self-negate pair $s = s^*$.

THEOREM 19. *Assume L is a totally ordered lattice with an order-reversing involutionary negation. If a, $b \in L$ and if $a \leqslant b$ then bFa in $\mathscr{L}$.*

Proof.

$$(b \wedge a) \wedge (b \vee a^*) = b \wedge a^* = b,$$
$$(b^* \vee a) \wedge (b^* \vee a^*) = b^* \wedge a^* = b^*.$$

Since DeMorgan's law is satisfied, this implies bfa in L. However, from Definition 6 this implies bFa in $\mathscr{L}$.

We now show that in a totally ordered lattice an appropriate negation leads to a total ordering of negate pairs with respect to fuzziness.

THEOREM 20. *Assume L is a totally ordered lattice with a negation having the involutionary and order-reversal properties. The relationship of fuzziness F, on the set $\mathscr{L}$ of negates leads to a total ordering over these negates in which for all a, $b \in \mathscr{L}$ if*

$$b \geqslant a \qquad \text{in } L$$

then

$$bFa \qquad \text{in } \mathscr{L}.$$

Proof. This follows directly from the above theorem and the partial ordering induced by an involution.

Thus, the relationship of fuzziness induces an ordering dependent upon the ordering in the set L.

As an immediate corollary to this theorem we can see the following:

COROLLARY 1. *If L is a totally ordered lattice with an order-reversing involutionary negation there exists a total ordering over the negate pairs (a, a^*), with respect to the fuzziness relationship F s.t.*

(1) *If $b^* \geqslant a^*$ (or equivalently, $a \leqslant b$) then aFb.*

(2) *If (a, a^*) is a self-negate, $a = a^*$, then this is a maximal fuzzy element.*

(3) *If L has a minimal element 0, and therefore a negate pair $(0, I)$, this pair is the least fuzzy.*

It should be noted that the three items listed in the above corollary correspond to a generalization in lattices of the three conditions stipulated by DeLuca and Termini (1972).

We have thus far specified our results in terms of the elements in the set of negate pairs $\mathscr{L}$. In anticipation of results for lattices which do not necessarily have the involution property and hence do not have unique negate pairs we shall reformulate our result in terms of fuzziness of elements in L.

THEOREM 21. *Assume L is a totally ordered lattice with a negation having involution and order reversal. The relationship of fuzziness, f on the set of L, leads* *to a total ordering over these elements in which, for all $a, b \in L$, if $(b \vee b^*) \leqslant (a \vee a^*)$ (or $(b \wedge b^*) \geqslant (a \wedge a^*)$) then*

$$bfa \quad \text{in } L.$$

Proof. This follows from Theorem 20 and its corollary, when we recall the fact that we defined $a = (a \wedge a^*)$ and $a^* = (a \vee a^*)$.

From Theorem 21 we see that the concept of fuzziness in an ordered lattice is related to the union of an element and its negation. In particular, the larger this union is, the less fuzzy it is. If we recall that $a \vee a^* = I$ is called the law of the excluded middle, Theorem 21 implies that the closer a proposition and its negation are to satisfying the law of the excluded middle the less fuzzy they are.

Let us consider the situation in which the lattice is not necessarily totally ordered, but is distributive. We shall assume that the negation satisfies DeMorgan's laws and is involutionary. Thus, the lattice misses being a Boolean algebra only in that it is not necessarily complemented.

THEOREM 22. *If L is a distributive lattice which has a negation which satisfies DeMorgan's laws and which is an involution, then for any a and $b \in L$*

$$afb \quad \text{if} \quad a \wedge (b \vee b^*) = a \quad \text{and} \quad a^* \wedge (b \vee b^*) = a^*.$$

Proof. In general, afb if $(a \vee b) \wedge (a \vee b^*) = a = (a \wedge b) \vee (a \wedge b^*)$ and $(a^* \vee b) \wedge (a^* \vee b) = a^* = (a^* \wedge b) \vee (a^* \wedge b^*)$. From the distributive property these conditions become $a \vee (b \wedge b^*) = a = a \wedge (b \vee b^*)$ and $a^* \vee (b \wedge b^*) = a = a \wedge (b \wedge b^*)$. However, if $a \wedge (b \vee b^*) = a$, then $(a \wedge (b \vee b^*))^* = a^*$, and from DeMorgan's law $a^* = a^* \vee (b^* \wedge b^{**})$; then involution implies $a^* = a^* \vee (b^* \wedge b)$. Similarly, $a^* \wedge (b \vee b^*) = a^*$ implies $a = a \vee (b \wedge b^*)$.

It is immediately obvious that under the conditions of this theorem $a \vee (b \wedge b^*) = a$ and $a^* \vee (b \wedge b^*) = a^*$ are equally valid for proving afb.

Before proceeding we note than in any lattice L, $x \vee y \geqslant x$. This follows from $x \vee (x \vee y) = (x \vee x) vy = xvy$.

THEOREM 23. *If L is a distributive lattice which has a negation which satisfies DeMorgan's laws and is involutary*

(1) $(a \vee a^*) \geqslant (b \vee b^*) \rightarrow bfa$,

(2) $(a \vee a^*) > (b \vee b^*) \rightarrow afb$.

Proof. (1) $(a \vee a^*) \geqslant (b \vee b^*) \geqslant b \Rightarrow (a \vee a^*) \geqslant b$ and therefore $b \wedge (a \vee a^*) = b$. Similarly, $b^* \wedge (a \vee a^*) = b^*$. Therefore, bfa.

(2) $afb \rightarrow a^* \wedge (b \vee b^*) = a^*$ and $a \wedge (b \vee b^*) = a$.

Therefore:

$$a \vee a^* = (a^* \vee (b \vee b^*)) \vee (a \wedge (b \vee b^*))$$
$$= a^* \vee (a \wedge (b \vee b^*)) \wedge ((b \wedge b^*) \vee (a \wedge (b \vee b^*)).$$

Since from absorption $(b \vee b^*) \vee (a \wedge (b \vee b^*)) = b \vee b^*$ we get $a \vee a^* = (a^* \vee a) \wedge (a^* \vee (b \vee b^*) \wedge (b \vee b^*)$. Since $(a^* \vee a) > (b \vee b^*)$ then $(b \vee b^*) \wedge (a^* \vee a^*) = b \vee b^* \; a \vee a^* = (b \vee b^*) \wedge (a^* \wedge (b \vee b^*)) = (b \vee b^*)$ by absorption; however, this contradicts our hypothesis that $(a \vee a^*) > (b \vee b^*)$.

Therefore, in a distributive lattice with a negation satisfying DeMorgan's laws and involution the measure of fuzziness of an element a is ordered by $a \vee a^*$.

This theorem then implies that the relative or ordinal fuzziness of an element in a distributive lattice with an involutionary negation satisfying DeMorgan's laws is determined by the union of the element with its negation. We note that the larger this intersection, the closer to I, the less fuzzy the element. In particular, we note that if $a \vee a^* = I$, then a is a least-fuzzy element. Recalling that an element satisfying $a \vee a^* = I$ is said to satisfy the law of the excluded middle we can say that the closer the element is to satisfying the law of the excluded middle the less fuzzy it is. In addition, if $a \vee a^* = b \vee b^*$ for all elements in the lattice then there exists no concept of fuzziness in the lattice.

Furthermore, we note that for a lattice having this type of negation $a \vee a^* = I$ implies $a \wedge a^* = 0$. That is, the law of the excluded middle implies the law of contradiction. In addition, if $a \vee a^* \geqslant b \vee b^*$ then $b \wedge b^* \geqslant a \wedge a^*$. This follows since if $a \vee a^* \geqslant b \vee b^*$, then $(a \vee a^*)^* \leqslant (b \vee b^*)^*$ and $a^* \wedge a \leqslant b^* \wedge b$. Thus, in a lattice with this type of negation the law of the excluded middle and the law of contradiction are equivalent. This implies that either closeness of $a \wedge a^*$ to 0 or of $a \vee a^*$ to I are both good measures of fuzziness.

The Intuitionistic Negation

An alternative approach to negation is possible based upon the pseudo complement in a Heything algebra (Goguen, 1969; Birkoff, 1968; Heything, 1956; Boyd, 1978).

Definition 9. A Heything algebra H is a lattice in which for any given elements a and b, the set of all $x \in H$ s.t. $a \wedge x \leqslant b$ contains a greatest element, called the relative pseudo complement of a in b. This element is denoted $a \rightarrow b$ and reads if a then b. The operation $a \rightarrow b$ is a binary operation in the lattice.

Note 1. Any Heything algebra is distributive.

Note 2. Any complete lattice is a Heything algebra iff the join operation is completely distributive on meets, that is,

$$a \wedge \left(\bigvee_{\alpha} X_{\alpha}\right) = \bigvee_{\alpha} (a \wedge x_{\alpha}) \qquad \text{for any set } \{x_{\alpha}\}.$$

Note 3. Boolean algebras, totally ordered lattices, and finite distributive lattices are Heything algebras.

DEFINITION 10. In a Heything algebra with a minimal element 0, the element $a \to 0$ is called the pseudo complement of a and is denoted $\bar{a}$. It is called the intuitionist negation.

Thus, in the following we shall assume H is a Heything algebra with minimal and maximal elements denoted by 0 and I, and we shall denote $\bar{a}$ as the negation. Thus, $\bar{a} = \vee\{x/a \wedge x = 0\}$.

THEOREM 24. *The following is a list of properties for the negation in a Heything algebra.*

H(1) $a \leqslant b \Rightarrow \bar{b} \leqslant \bar{a}$.

H(2) $a \leqslant \bar{\bar{a}}$.

H(3) $\bar{a} = \bar{\bar{\bar{a}}}$.

H(4) $\overline{(a \vee b)} = \bar{a} \wedge \bar{b}$.

H(5) $\overline{(a \wedge b)} \geqslant \bar{a} \vee \bar{b}$.

H(6) In a Heything algebra if x has a complement it must be $\bar{x}$.

H(7) $\bar{I} = 0$ and $\bar{0} = I$.

H(8) $z \leqslant \bar{x}$ iff $z \wedge x = 0$.

Note that the law of double negation does not necessarily hold; that is, $(\bar{\bar{a}}) \neq a$. Thus, in a Heything algebra negation is not an involution. It can be shown that if negation is an involution, then the Heything algebra is a Boolean algebra.

THEOREM 25. *If H is a completely distributive Heything algebra then $a \wedge \bar{a} = 0$.*

Proof.

$$\bar{a} = \bigvee_{\alpha} \{x_{\alpha}/x_{\alpha} \wedge a = 0\},$$

$$a \wedge \bar{a} = a \wedge \bigvee_{\alpha} x_{\alpha} = \bigvee_{\alpha} a \wedge x_{\alpha} = \bigvee_{\alpha} 0 = 0.$$

Thus, the law of contradiction holds in Heything algebra for all elements. However, we note that, in general, the law of the excluded middle does not always hold. That is, $a \vee \bar{a} \neq 1$. This implies that there is not the tight relationship between the law of contradiction and the law of the excluded middle which exists with involutionary order-reversing negations.

To get a better understanding of what the Heything negation implies we recall that in a distributive lattice we say b lies between a and c, denoted by (a, b, c), if

$$b \wedge (a \vee c) = b = b \vee (a \wedge c).$$

Using this we prove a theorem.

THEOREM 26. *Assume H is a complete Heything algebra. If $y \in H$ is s.t. $a \wedge y = 0$ then y lies between a and $\bar{a}$.*

Proof. $y \wedge (a \vee \bar{a}) = (y \wedge a) \vee (y \wedge \bar{a})$, from H8 we see that $(y \wedge a) \vee (y \wedge \bar{a}) = (y \wedge a) \vee y$, then from absorption we see $(y \wedge a) \vee y = y$. $y \vee (a \wedge \bar{a}) = y \vee 0 = y$.

Thus, we see that the intuitionistic negation of an element a in the lattice is an element which is disjoint from a, $a \wedge \bar{a} = 0$, such that every other disjoint element lies between it and a. That is, it is the disjoint element farthest from a.

We also note that in a Heything algebra and element cannot be self-negate.

THEOREM 27. *In a Heything algebra $a \neq \bar{a}$.*

Proof. Assume $a = \bar{a}$, from the previous theorem, $a \wedge \bar{a} = 0$; therefore, $a \wedge a = 0 \Rightarrow a = 0$. However, we know $\bar{a} = I$ if $a = 0$. Thus, we have a contradiction if $a = \bar{a}$.

Since the intuitionistic negation does not in general satisfy the property of involution we cannot talk about a unique decomposition along negate pairs. However, the join of an element and its negation still play a significant role in discussing the concept of fuzziness of elements in H.

THEOREM 28. *Assume H is a complete Heything algebra with intuitionistic negation, then for any $x, y \in H$*

$$xfy \quad \text{if} \quad x \wedge (y \vee \bar{y}) = x \quad \text{and} \quad \bar{x} \wedge (y \vee \bar{y}) = x.$$

Proof. Since any Heything algebra is distributive we can use the distributive form of betweenness. Therefore, the conditions for xfy become

(1) $x \vee (y \wedge \bar{y}) = x$,

(2) $x \wedge (y \vee \bar{y}) = x$,

(3) $\bar{x} \vee (y \wedge \bar{y}) = \bar{x}$,

(4) $\bar{x} \wedge (y \vee \bar{y}) = \bar{x}$.

However, in a Heything algebra $y \wedge \bar{y} = 0$. Thus, conditions (1) and (3) are always satisfied. Therefore, xfy if conditions (2) and (4) are satisfied.

THEOREM 29. *In a complete Heything algebra H, for any two elements $x, y \in H$*

(1) $x \vee \bar{x} \geqslant y \vee \bar{y} \Rightarrow yfx.$

(2) $x \vee x > y \vee \bar{y} \Rightarrow xfy.$

Proof. (1) Assume $x \vee \bar{x} \geqslant y \vee \bar{y}$; then $x \vee \bar{x} \geqslant y$ and $x \vee \bar{x} \geqslant \bar{y}$. This implies $y \wedge (x \vee \bar{x}) = y$ and $\bar{y} \wedge (x \vee \bar{x}) = \bar{y}$. From the previous theorem this implies yfx.

(2) If xfy then $x \wedge (y \vee \bar{y}) = x$ and $\bar{x} \wedge (y \vee y) = \bar{x}$; this implies $x \vee \bar{x} = (x \wedge (y \vee \bar{y}) \vee (\bar{x} \wedge (y \vee \bar{y}) = (x \vee (\bar{x} \wedge (y \vee \bar{y})) \wedge ((y \vee \bar{y}) \vee (x \wedge (y \vee \bar{y}))$. However, from absorption $(y \vee \bar{y}) \vee (\bar{x} \wedge (y \vee \bar{y}) = y \vee \bar{y}$; therefore, $x \vee \bar{x} = (x \vee (\bar{x} \wedge (y \vee \bar{y}))) \wedge (y \vee \bar{y}) = ((x \vee \bar{x}) \wedge (x \vee (y \vee \bar{y})) \wedge (y \vee \bar{y}) = (x \vee \bar{x}) \wedge (y \vee \bar{y}) = y \vee \bar{y}$, which implies $y \vee \bar{y} = x \vee \bar{x}$, which contradicts our hypothesis.

THEOREM 30. *In a complete Heything algebra the relationship of fuzziness f, defined on H, is a partial ordering such that*

$$yfx \quad \text{if} \quad x \vee \bar{x} \geqslant y \vee \bar{y}.$$

COROLLARY. *If in a Heything algebra, $x \vee \bar{x}$ and $y \vee \bar{y}$ are comparable for all $x, y \in H$, then f is a total ordering.*

Thus, as in the case of a distributive lattice having a negation which is involutionary and satisfying DeMorgan's laws, a Heything algebra with an tuitionistic negation has a relationship of fuzziness determined by the join of an element and its negation. In particular, the larger $a \vee \bar{a}$, the less fuzzy the element a. In either case, if a is equal to a maximal element it is least fuzzy. That is, the closet $a \vee \bar{a}$, is to I, the less fuzzy it is. Thus, in both cases the fuzziness of an element is determined by the satisfaction to the law of the excluded middle.

However, whereas with the involutionary-type negation the intersection of an element with its negation is also a valid measure of fuzziness, it is not so with the intuitionistic negation. In this case, the law of the contradiction is always satisfied and does not help.

EXAMPLE. Assume H is a totally ordered lattice and is therefore a Heything algebra. Since $x \wedge \bar{x} = \text{Min}(x, \bar{x})$ and $x \wedge \bar{x} = 0$ with a Heything negation, this implies that if $x \neq 0$ then $\bar{x} = 0$. Thus, in a totally ordered Heything algebra the Heything negation of any element other then the minimal element is the minimal element. Since the negation of the minimal element is the maximal element we get

$$\bar{x} = 0 \quad \text{if} \quad x \neq 0,$$

$$\bar{x} = 1 \quad \text{if} \quad x = 0.$$

Therefore

$$x \vee \bar{x} = x \quad \text{if} \quad x \neq 0,$$

$$x \vee \bar{x} = I \quad \text{if} \quad x = 0.$$

Therefore, in a totally ordered Heything algebra the relationship f defined on H is a total ordering s.t.

$$xf0, \qquad \text{and} \qquad xfy \qquad \text{if} \quad y > x > 0.$$

Thus, for $x \neq 0$, the farther x is from the minimal element the less fuzzy x is.

Comparing Negations

In order to compare these different forms of negation we shall borrow a concept from Rescher (1969).

Definition 10. Assume L is a lattice with universal bounds and a negation. The strength of the negation is defined as the proportion of the elements in L for which $x \wedge x^* = 0$. That is,

$$S = \frac{|V|}{|L|},$$

where $V = \{x/x \wedge x^* = 0\}$ and $|V|$ and $|L|$ are the cardinalities of the respective sets. It is noted that $0 \leqslant S \leqslant 1$, where the larger the S, the stronger the negation.

Theorem 31. *In a totally ordered lattice:*

(1) *An intuitionistic negation has* $S = 1$.

(2) *An involutionary negation has* $S = 2/|L|$.

Proof. (1) This follows from the fact that $x \wedge x^* = 0$ for any Heything negation.

(2) From total ordering $x \wedge x^* \in \{x, x^*\}$; therefore $x \wedge x^* = 0$ implies $0 \in \{x, x^*\}$.

Furthermore, since in an involutionary negation (x, x^*) form unique pairs, one and only one element can have 0 as its negation, that is, I. Similarly, 0 has I as its negation, so that I and 0 are only elements s.t. $x \wedge x^* = 0$.

Thus, we see that the Heything negation is a very strong negation while an involutary negation is weak. For example, if we use the elements of the lattice to indicate the truth values in a multivalued logic then an intuitionistic negation makes false the negation of any statement that is partially true. That is, if p is any proposition having any degree of truth, not p, using the intuitionistic negation, is completely false.

It should be noted that Zadeh's form of negation for $L = [0, 1]$, $a^* = 1 - a$ is involutionary and therefore very weak in the sense of our definition.

An alternative negation which may be useful in some situations is a threshold-type negation. If L is a totally ordered lattice, we could define

$$\begin{array}{lll} x^* = 0 & \text{if} \quad x > \alpha & \\ & & \alpha \in L. \\ x^* = 1 & \text{if} \quad x \leqslant \alpha & \end{array}$$

This negation is order reversing but not involuntary. We note in this case that

$$\begin{array}{lll} x > \alpha; & x \vee x^* = x, & x \wedge x^* = 0, \\ x \leqslant \alpha; & x \vee x^* = 1, & x \wedge x^* = x, \end{array}$$

so that either the law of the excluded middle or the law of contradiction is satisfied. With this type of negation fuzziness is not uniquely described by $a \vee a^*$. It requires knowledge of both $a \vee a^*$ and $a \wedge a^*$.

An interesting property of the intuitionistic negation is that it operates as a defuzzifier. That is, x is at least as fuzzy as $\bar{x}$. This is in contrast to the involutionary case in which x and its negation are equally fuzzy.

THEOREM 32. *If negation is intuitionistic in a Heything algebra then* $xf\bar{x}$ *for all* x.

Proof. Recalling that from H2, $x \leqslant x$, this implies $\ddot{x} \vee \bar{x} \geqslant \bar{x} \vee x$, which implies $xf\bar{x}$.

This is an interesting and perhaps useful property at times. In a sense, this property implies that the negation is moving the truth values to the extreme points. Zadeh's form of negation used in the unit interval doesn't have this property. It may be desirable in some situations to have a negation on the unit interval which acts in this defuzzifying manner.

With these ideas in mind we may conjecture a general class of negations for the unit interval lattice, [0, 1].

Assume

$$0 \leqslant a \leqslant b \leqslant c \leqslant 1 \qquad \text{and} \qquad n \geqslant 1;$$

then we can define a general negation as: N: $[0, 1] \to [0, 1]$, where $N(v) = v^*$, s.t.

$$
\begin{aligned}
v^* &= 1, && 0 \leqslant v < a, \\
v^* &= 1 - \frac{1}{2}\left[\frac{V - a}{b - a}\right]^n, && a \leqslant v < b, \\
v^* &= \frac{1}{2}\left[\frac{c - V}{c - b}\right]^n, && b \leqslant v < c, \\
v^* &= 0, && v \geqslant c.
\end{aligned}
$$

Figure 1 describes its general shape.

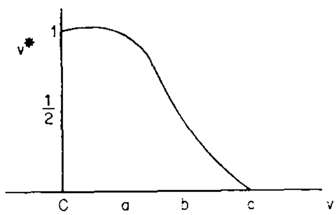

FIG. 1. General negation.

We note the following:

(1) $a = 0$, $b = \frac{1}{2}$, $c = 1$, and $n = 1$. We get Zadeh's negation

$$v^* = 1 - v.$$

(2) If $n \to \infty$, then we get a threshold-type negation. In particular, we get:

$$
\begin{aligned}
v^* &= 1, && 0 \leqslant v < b, \\
v^* &= 0, && v > b.
\end{aligned}
$$

When $b \to 0$ in this case, we get the intuitionistic negation.

We also note that this negation is order reversing. We leave the investigation of the properties of this negation for a further study.

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